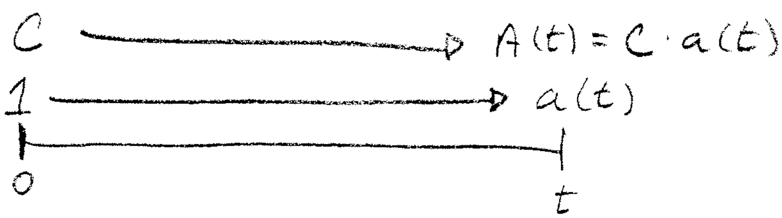
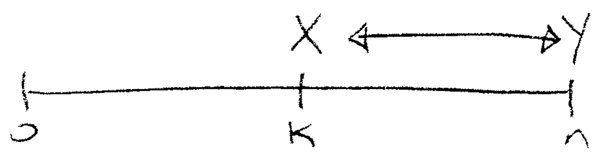
Module 1

Section 1: Accumulation Functions

The **accumulation function**, denoted $a(t)$, gives the value at time t for an initial time 0 investment of 1. So $a(0) = 1$. The **amount function**, denoted $A(t)$, gives the value at time t for an initial time 0 investment of C. So $A(0) = C$. The amount function is obtained from the accumulation function by multiplying the accumulation function by the amount of the initial investment. The "timeline" is:



We focus primarily on the accumulation function. We **accumulate** from time 0 to time t by multiplying by $a(t)$, and we **discount** from time t back to time 0 by dividing by $a(t)$. We can relate a time k value of X to its equivalent (or indifference) time n value Y using accumulation functions as follows:



$$Y = X \cdot \frac{a(n)}{a(k)}\text{, or solving for } X \text{ we get, } X = Y \cdot \frac{a(k)}{a(n)}$$

For each deposit, the **amount of interest** earned between times k and n equals the difference between the equivalent time n value of the deposit and the equivalent time k value of the deposit.

Module 1 Section 1 Problems:

Note: The problems at the end of each section are "warm-up" exercises. These are generally **not** the type of problem that you will see on an actuarial exam. Actuarial exam type problems are generally harder problems that cover more than what is covered in any one section. These types of problems are at the end of each module. As with all math problems, strive to use correct notation.

For Problems 1-8, you are given the following accumulation function information:

$$a(1) = 1.2,\ a(2) = 1.5,\ a(3) = 2.0, \text{ and } a(4) = 3.0$$

(Remember that $a(0) = 1$ for all accumulation functions.)

1. 100 is deposited at time $t = 0$. Determine the accumulated amount at time $t = 3$.

2. Determine the present value at time $t = 0$ of 60 at time $t = 4$.

3. The value at time $t = 2$ is 300. Determine the accumulated value at time $t = 3$.

4. Determine the discounted value at time $t = 1$ of a value of 600 at time $t = 4$.

5. Given 480 at time $t = 1$, plus 300 at time $t = 3$:

    a. Determine the (total) present value at time $t = 0$ of the two payments.
    b. Determine the (total) accumulated value at time $t = 4$ of the payments.
    c. Determine the (total) value at time $t = 2$ of the payments

6. Determine the amount of interest earned from time $t = 2$ to time $t = 4$ if 500 is invested at time $t = 0$.

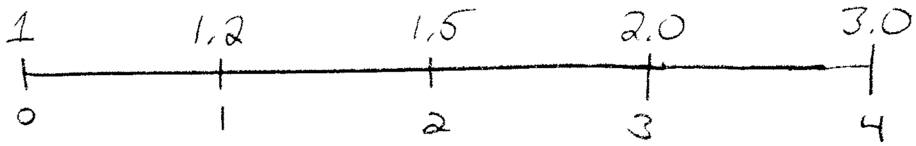
7. Determine the amount of interest earned from time $t = 2$ to time $t = 3$ if 300 is invested at time $t = 1$.

8. Determine the amount of interest earned from time $t = 2$ to time $t = 4$ if 240 is invested at time $t =1$ and an additional 300 is invested at time $t = 3$.

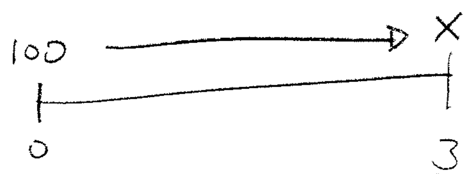
Solutions to Module 1 Section 1 Problems

1-8:

Set-up



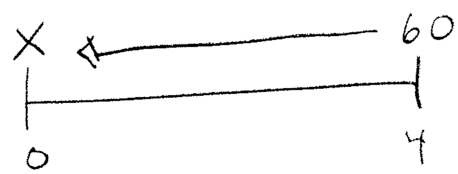
1)



$$X = 100 \cdot a(3) = 100 \cdot (2{,}0)$$

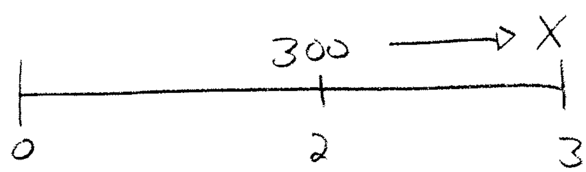
$\therefore \boxed{X = 200}$

2)



$$X = \frac{60}{a(4)} = \frac{60}{3{,}0} \Longrightarrow \boxed{X = 20}$$

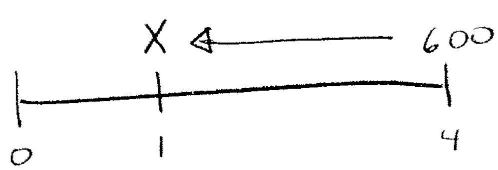
3)



$$X = 300 \cdot \frac{a(3)}{a(2)} = 300 \frac{2{,}0}{1{,}5}$$

$\therefore \boxed{X = 400}$

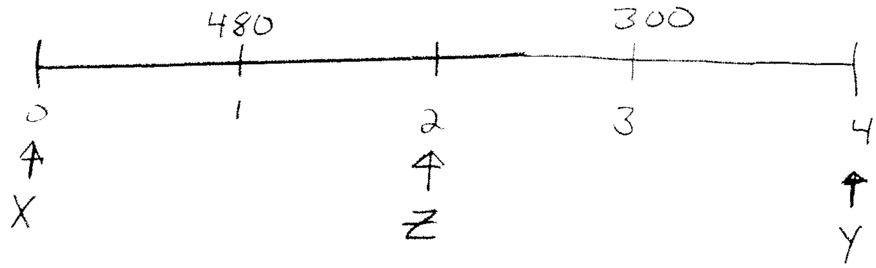
4)



$$X = 600 \frac{a(1)}{a(4)} = 600 \frac{1{,}2}{3{,}0}$$

$\therefore \boxed{X = 240}$

5)



(a) $X = \frac{480}{a(1)} + \frac{300}{a(3)} = \frac{480}{1.2} + \frac{300}{2.0}$

$\therefore \boxed{X = 550}$

(b) $Y = 480 \frac{a(4)}{a(1)} + 300 \frac{a(4)}{a(3)}$

$= 480 \frac{3.0}{1.2} + 300 \frac{3.0}{2.0} \Rightarrow \boxed{Y = 1650}$

(c) $Z = 480 \frac{a(2)}{a(1)} + 300 \frac{a(2)}{a(3)}$

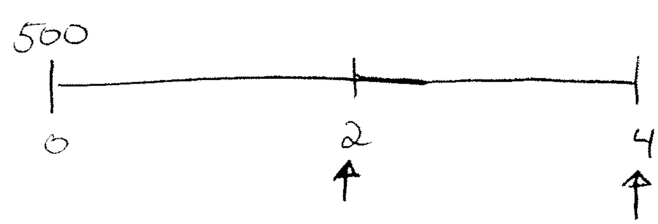
$= 480 \frac{1.5}{1.2} + 300 \frac{1.5}{2.0} \Rightarrow \boxed{Z = 825}$

Note: Once we found X, we could have determined Y & Z using the facts

1) $Y = X \cdot a(4)$, and

2) $Z = X \cdot a(2)$.

6)

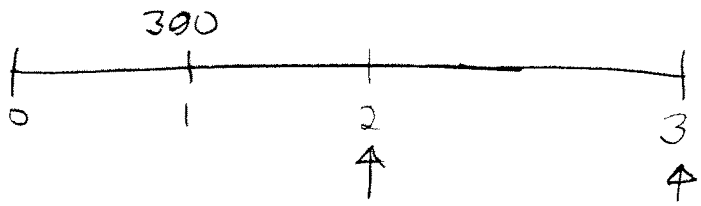


$I_{[2,4]} = 500 \cdot a(4) - 500 \cdot a(2)$

$= 500(3.0) - 500(1.5)$

$\therefore \boxed{I_{[2,4]} = 750}$

7)

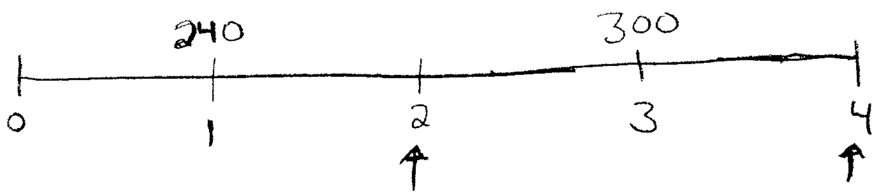


$$I_{[2,3]} = 300\,\frac{a(3)}{a(1)} - 300\,\frac{a(2)}{a(1)}$$

$$= 300\left(\frac{2.0}{1.2}\right) - 300\left(\frac{1.5}{1.2}\right)$$

$$\therefore \boxed{I_{[2,3]} = 125}$$

8)



For the 240 payment, we get the interest amount from time $t=2$ to $t=4$ is $I_1 = 240\,\frac{a(4)}{a(1)} - 240\,\frac{a(2)}{a(1)}$

$$\therefore I_1 = 240\left(\frac{3.0}{1.2}\right) - 240\,\frac{(1.5)}{(1.2)} = 300$$

For the 300 payment, interest is earned from time $t=3$ to $t=4$ and equals $I_2 = 300\,\frac{a(4)}{a(3)} - 300 = 300\left(\frac{3}{2}\right) - 300$

$$\therefore I_2 = 150$$

$$\therefore I_{[2,4]} = 300 + 150 = 450$$

(Alternate Solution): The total accumulated value of the payments at time 4 is $AV_4 = 240\,\frac{a(4)}{a(1)} + 300\,\frac{a(4)}{a(3)} = 240\left(\frac{3.0}{1.2}\right) + 300\left(\frac{3.0}{2.0}\right)$

$$\therefore AV_4 = 1050$$

At time 2, we only accumulate the 240 payment, getting

$$AV_2 = 240\,\frac{a(2)}{a(1)} = 240\left(\frac{1.5}{1.2}\right) = 300$$

The difference, $AV_4 - AV_2 = 1050 - 300 = 750$ is accounted for by interest and additional payments. The additional payment is 300 (at $t=3$) and so the interest is $I_{[2,4]} = 750 - 300 = 450$.

Section 2: Simple and Compound Interest

Simple Interest:

$a(t) = 1 + it$, where i is the simple interest rate and t is measured in years

Discrete Compound Interest: (Converted and Payable are synonyms for Compounded)

$a(t) = (1 + i)^t$, where i is the periodic effective interest rate (eir) and t is measured in same time unit (match periods for i and t).

In the context of discrete compounding, $1 + i$ is the periodic accumulation factor and $v = \frac{1}{1+i}$ is the periodic discount factor.

Continuously Compounding Interest:

$a(t) = e^{\delta t}$, where δ is the continuously compounded interest rate and t is measured in years. Actuaries refer to δ as the (constant) force of interest.

In the context of continuous compounding, e^{δ} is the annual accumulation factor and $v = e^{-\delta}$ is the annual discount factor.

Periodic Effective Interest Rates:

$i_k = \frac{a(k)-a(k-1)}{a(k-1)}$ is the periodic effective interest rate (eir) for the k^{th} period.

In the context of compounding, i_k is constant. We abbreviate the monthly effective interest rate by meir, the quarterly effective interest rate is abbreviated by qeir, etc.

Nominal Interest Rates:
In the context of discrete compounding, we let m denote the number of compounding periods per year. The nominal interest rate is the rate quoted and is denoted by $i^{(m)}$. The periodic eir is $i = \frac{i^{(m)}}{m}$.

Equivalent Rates: (Indifference Rates)

When compounding, we determine equivalent rates by accumulating or discounting a given amount (we can use $1) over an arbitrary period of time. For simple interest, we must be given the period of time over which to accumulate or discount.

Module 1 Section 2 Problems:

1. An account pays 3% simple interest.
    a. Determine the accumulation function.
    b. Determine the effective interest rate for the 4th year.
    c. Determine the effective interest rate for the 6th year.
    d. Determine the effective interest rate for the 9th month.
    e. Determine the effective interest rate for the 13th month.

2. Four years ago David made an initial deposit into an account that pays 6% simple interest. Unfortunately, David does not remember how much he initially deposited into the account. He currently has 1240 in the account. Determine how much David will have in the account one year from now.

3. An account pays 6% interest, compounded monthly.
    a. Determine the accumulation function.
    b. Determine the effective interest rate for the 3rd month.
    c. Determine the effective interest rate for the 5th month.
    d. Determine the monthly accumulation factor.
    e. Determine the monthly discount factor.
    f. Determine the effective interest rate for the 2nd quarter.
    g. Determine the effective interest rate for the 4th quarter.
    h. Determine the quarterly accumulation and discount factors.

4. Four years ago David made an initial deposit into an account that pays 6% interest, compounded semiannually. Unfortunately, David does not remember how much he initially deposited into the account. He currently has 1240 in the account. Determine how much David will have in the account one year from now.

5. An account pays interest using a constant force of interest equal to 7%.
    a. Determine the accumulation function.
    b. Determine the annual accumulation and discount factors and the aeir.
    c. Determine the monthly accumulation and discount factors and the meir.

6. An account pays 7% interest compounded annually. Determine the equivalent force of interest.

7. An account pays 4% interest compounded semiannually. Determine the equivalent force of interest.

8. Four years ago David made an initial deposit into an account that pays interest using a constant force of interest equal to 6%. Unfortunately, David does not remember how much he initially deposited into the account. He currently has 1240 in the account. Determine how much David will have in the account one year from now.

9. X is deposited into an account that pays 3% interest, compounded quarterly. The accumulated value after 8 years is 10000. Determine X.

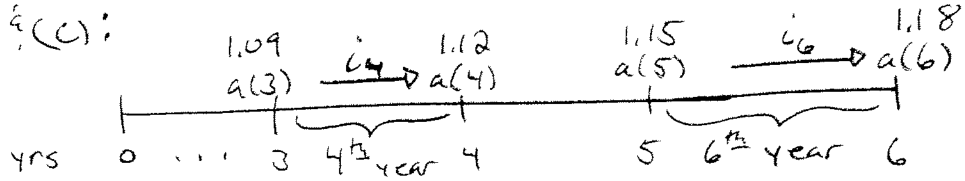
10. 200 is deposited into an account that pays 6% simple interest for the first 3 years the money is in the account, then 8% compounded quarterly for the next 2 years the money is in the account, then a constant force of interest of 5% thereafter. Determine the amount in the account after 10 years.

Solutions to Module 1 Section 2 Problems

1) $i = .03$ simple

(a): $\boxed{a(t) = 1 + .03t \quad t\text{- years}}$

(b) & (c):



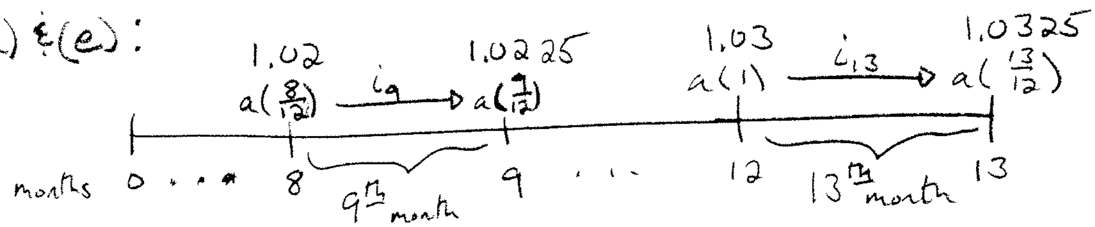
$1.09(1+i_4) = 1.12$

$\Rightarrow \boxed{i_4 = \frac{1.12}{1.09} - 1 \doteq 2.752\%}$

$1.15(1+i_6) = 1.18$

$\Rightarrow \boxed{i_6 = \frac{1.18}{1.15} - 1 \doteq 2.609\%}$

(d) & (e):



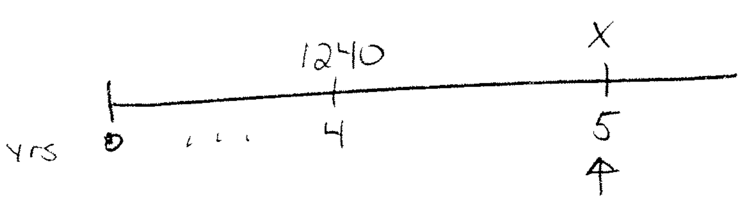
$1.02(1+i_9) = 1.0225$

$\Rightarrow$ $i_9 \doteq 0.2451\%$ eir for 9^{th} month

$1.03(1+i_{13}) = 1.0325$

$\Rightarrow$ $i_{13} \doteq 0.2427\%$ eir for 13^{th} month

2) $i = .06$ simple



$$X = 1240 \frac{a(5)}{a(4)}$$

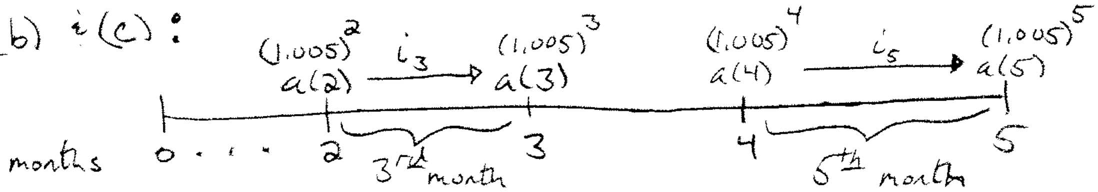
$a(t) = 1 + .06t \Rightarrow a(4) = 1.24$ and $a(5) = 1.30$

$$\therefore \boxed{X = 1240 \frac{1.3}{1.24} = 1300}$$

3) $i^{(12)} = .06 \implies \frac{i^{(12)}}{12} = \frac{.06}{12} = .005 =$ monthly eir (meir)

(a): $\boxed{a(t) = (1.005)^t}$ t - months

(b) & (c):



$(1.005)^2(1+i_3) = (1.005)^3$

$\implies \boxed{i_3 = \frac{(1.005)^{\cancel{3}}}{\cancel{(1.005)^2}} - 1 = .005}$

$(1.005)^4(1+i_5) = (1.005)^5$

$\implies \boxed{i_5 = \frac{(1.005)^{\cancel{5}}}{\cancel{(1.005)^4}} - 1 = .005}$

Note: When compounding, the periodic effective interest rate (eir) is constant for each period. This is the reason for the terminology and notation in the first line above. That is, given $i^{(12)}$, then $i = \frac{i^{(12)}}{12} =$ meir.

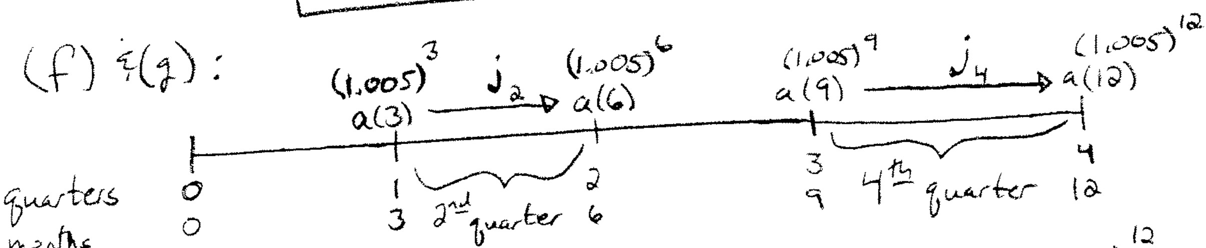
(d): $i^{(12)} = .06 \implies i = \frac{.06}{12} = .005 =$ meir

$\therefore$ $\boxed{\text{monthly accumulation factor} = 1+i = 1.005}$ "maf"

(e): monthly discount factor $= \dfrac{1}{\text{monthly accumulation factor}}$

$\boxed{mdf = v = \frac{1}{1.005}}$ Sometimes we write $v_{.005}$ to emphasize $i = .005$.

(f) & (g):



$(1.005)^3(1+j_2) = (1.005)^6$

$\implies j_2 = (1.005)^3 - 1 \doteq .01508$

$(1.005)^9(1+j_4) = (1.005)^{12}$

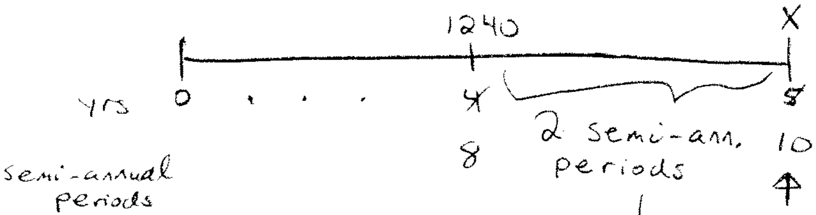
$\implies j_4 = (1.005)^3 - 1 = j_2$ ✓ to be expected since compounding (see above note)

The qeir equivalent to an meir of $i = .05$ is $\boxed{j = (1.005)^3 - 1 \doteq .01508}$

(h): $\boxed{qaf = 1+j \doteq 1.01508}$ and $\boxed{qdf = v_j = \frac{1}{1+j} \doteq \frac{1}{1.01508}}$

4) $i^{(2)} = .06 \implies i = \frac{.06}{2} = .03 = \text{seir}$

$\therefore a(t) = (1.03)^t$ t - semiannual periods



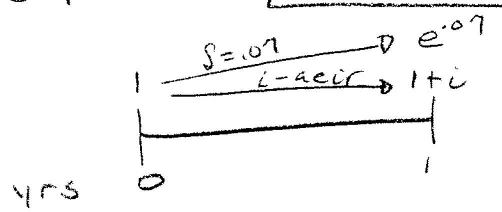
$$X = 1240\,\frac{a(10)}{a(8)} = 1240\,\frac{(1.03)^{10}}{(1.03)^8}$$

$$\therefore X = 1240(1.03)^2 \implies \boxed{X \doteq 1315.52}$$

<u>Note</u>: This example illustrates the fact that <u>when</u> <u>compounding</u>, our answers will not depend on the times of the payments but rather the number of periods between the payments and the valuation date.

5) $\delta = .07 \implies \boxed{a(t) = e^{.07t} \quad t\text{- years}}$

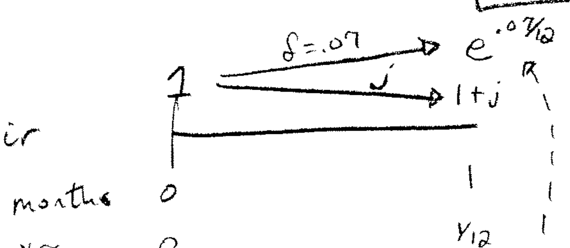
(b):



} set these expressions equal

$$\therefore \boxed{\begin{aligned} 1+i &= e^{.07} = \text{aaf} \\ \frac{1}{1+i} &= e^{-.07} = v_i = \text{adf} \\ i &= e^{.07} - 1 = \text{aeir} \end{aligned}}$$

(c) $j = \text{meir}$

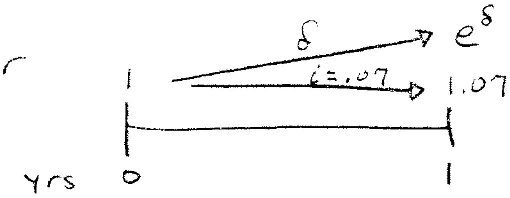


} set these expressions equal

$$\therefore \boxed{\begin{aligned} 1+j &= e^{.07/12} = \text{maf} \\ \frac{1}{1+j} &= e^{-\frac{.07}{12}} = v_j = \text{mdf} \\ j &= e^{.07/12} - 1 = \text{meir} \end{aligned}}$$

<u>Remark</u>: with $\delta = .07$, $a(\tfrac{1}{12}) = e^{.07(\frac{1}{12})} = e^{\frac{.07}{12}}$

6) $i = .07 = \text{aeir}$

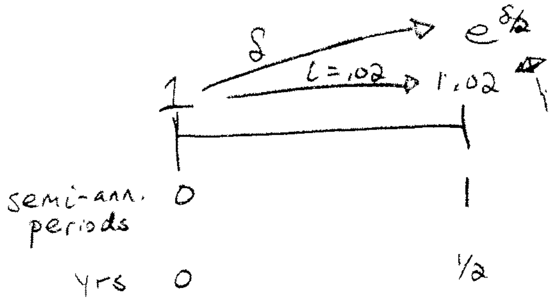


$\Big\} \implies e^{\delta} = 1 + i = 1.07$

$\implies \boxed{\delta = \ln(1.07)}$

7) $i^{(2)} = .04 \implies i = \frac{.04}{2} = .02 = \text{seir}$

$\therefore a(t) = (1.02)^t$ $\quad$ t - semiannual periods



$\therefore e^{\delta/2} = 1.02$

$\implies \frac{\delta}{2} = \ln(1.02)$

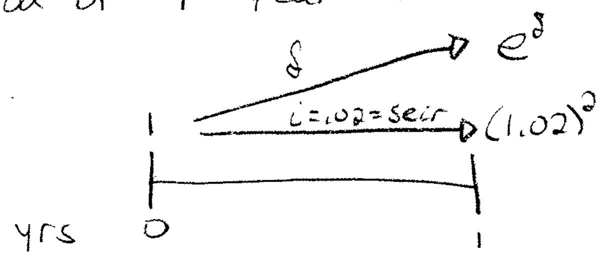
$\implies \boxed{\delta = 2\ln(1.02)}$

Remark: For δ, $a(t) = e^{\delta t}$ t-yrs

$\implies a(1/2) = e^{\delta/2}$

Note: $\delta = 2\ln(1.02) = \ln(1.02)^2$. The exponential form of the equation $\delta = \ln(1.02)^2$ is $e^{\delta} = (1.02)^2$. This is the equation we would have solved if we would have used a time period of 1 year instead of using 1 semi-ann. period above.

I.e.

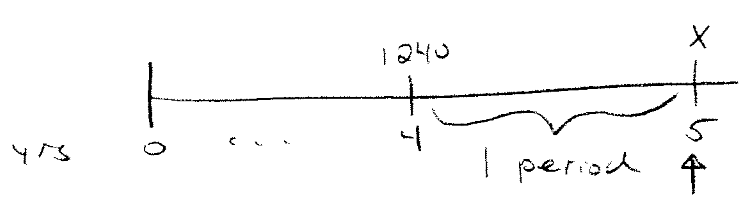


$\left.\right\} \implies e^{\delta} = (1.02)^2$

$\implies \delta = \ln(1.02)^2 = 2\ln(1.02)$ as above

This illustrates that the time period used for the accumulation (or discount) when determining equivalent rates is arbitrary when we're compounding.

8) $\delta = .06 \implies a(t) = e^{.06t}$ $\quad$ t- yrs (continuous compounding)

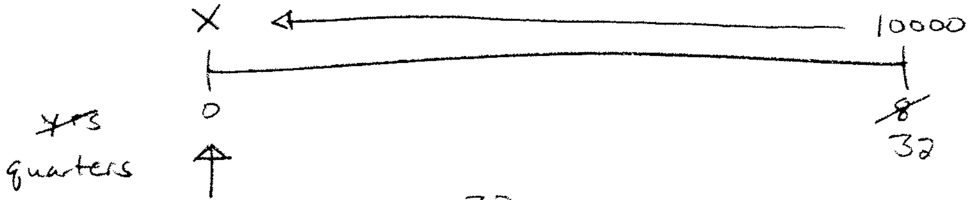


Since we're compounding, we can use the fact in the note in #4.

$$\boxed{X = 1240\, e^{.06(1)} \doteq 1316.68}$$

9) $i^{(4)} = .03 \implies i = \frac{.03}{4} = .0075 = \text{qeir}$

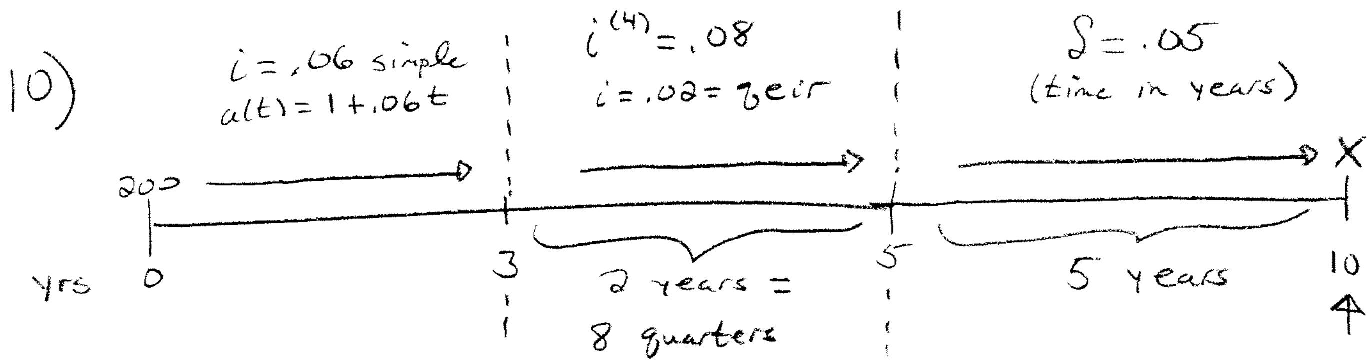
$\implies a(t) = (1.0075)^t$ $\quad$ t - quarters



$X = 10000\, v^{32}$ $\quad$ where $v = \text{qdf} = \frac{1}{1.0075}$

$\therefore \boxed{X \doteq 7873.33}$

10)



$\therefore X = 200(1 + .06(3)) \cdot (1.02)^8 \cdot e^{.05(5)}$

$\implies \boxed{X \doteq 355.05}$

Section 3: Simple and Compound Discount

Simple Discount:

$a(t) = (1 - dt)^{-1}$, where d is the simple discount rate and t is measured in years

Discrete Compound Discount: (Converted and Payable are synonyms for Compounded)

$\mathrm{a(t)} = (1 - d)^{-t}$, where d is the periodic effective discount rate (edr) and t is measured in same time unit (period).

In the context of discrete compounding, $(1 - d)^{-1}$ is the periodic accumulation factor and $v = 1 - d$ is the periodic discount factor.

Continuously Compounding Discount: (Same as continuously compounded interest.)

$a(t) = e^{\delta t}$, where δ is the continuously compounded discount rate and t is measured in years.

In the context of continuous compounding, e^{δ} is the annual accumulation factor and $v = e^{-\delta}$ is the annual discount factor.

Periodic Effective Discount Rates:

$d_k = \frac{a(k) - a(k-1)}{a(k)}$ is the periodic effective discount rate (edr) for the k^{th} period.

In the context of compounding, d_k is constant. We abbreviate the monthly effective discount rate by medr, the quarterly effective discount rate is abbreviated by qedr, etc.

Nominal Discount Rates:
In the context of discrete compounding, we let m denote the number of compounding periods per year. The nominal discount rate is the rate quoted and is denoted by $d^{(m)}$. The periodic edr is $d = \frac{d^{(m)}}{m}$.

Equivalent Rates: (Indifference Rates) (Same as with interest rates.)

When compounding, we determine equivalent rates by accumulating or discounting a given amount (we can use $1) over an arbitrary period of time. For simple discount, we must be given the period of time over which to accumulate or discount.

Module 1 Section 3 Problems:

1. An account pays interest using a 5% simple discount rate.
   a. Determine the accumulation function.
   b. Determine the effective interest rate for the 4^{th} year.
   c. Determine the effective discount rate for the 4^{th} year.
   d. Determine the effective interest rate for the 6^{th} year.
   e. Determine the effective discount rate for the 6^{th} year.

2. Four years ago Carol made an initial deposit into an account that pays interest using a 6% simple discount rate. Unfortunately, Carol does not remember how much she initially deposited into the account. She currently has 1000 in the account. Determine how much Carol will have in the account one year from now.

3. An account pays interest using a 8% discount rate, compounded semiannually.
   a. Determine the accumulation function.
   b. Determine the semiannual discount and accumulation factors.
   c. Determine the equivalent aeir and aedr.

4. Four years ago Carol made an initial deposit into an account that pays interest using a 6% discount rate, compounded quarterly. Unfortunately, Carol does not remember how much she initially deposited into the account. She currently has 1000 in the account. Determine how much Carol will have in the account one year from now.

5. An account pays interest using a constant force of discount equal to 7%.
   a. Determine the accumulation function.
   b. Determine the annual accumulation and discount factors.
   c. Determine the equivalent aeir and aedr.

6. An account pays interest using a 7% discount rate, compounded annually. Determine the equivalent force of interest.

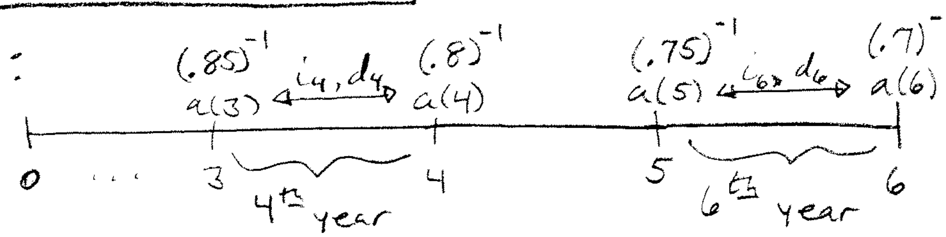
7. An account pays interest using a 4% discount rate, compounded semiannually. Determine the equivalent force of interest.

Solutions to Module 1 Section 3 Problems

1) $d = .05$ simple

(a) $\boxed{a(t) = (1 - .05t)^{-1}}$ t - years

(b) - (e):



In order to determine equivalent eir's and/or edr's we can accumulate or discount. It's easier to accumulate when determining eir's and to discount when determining edr's.

(b): Determine i_4: $a(3)(1+i_4) = a(4)$

$$\Rightarrow \boxed{i_4 = \frac{(.8)^{-1}}{(.85)^{-1}} - 1 = \frac{85}{80} - 1 = \frac{5}{80}}$$

(c) Determine d_4: $a(4)(1-d_4) = a(3)$

$$\Rightarrow \boxed{d_4 = 1 - \frac{(.85)^{-1}}{(.8)^{-1}} = 1 - \frac{80}{85} = \frac{5}{85}}$$

<u>Note</u>: Using accumulation, we would have $a(3)(1-d_4)^{-1} = a(4)$. We would get the same answer as above.

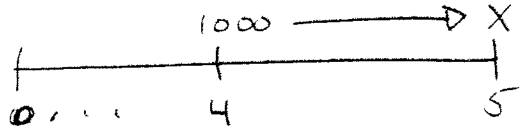
(d) Determine i_6: $a(5)(1+i_6) = a(6)$

$$\Rightarrow \boxed{i_6 = \frac{(.7)^{-1}}{(.75)^{-1}} - 1 = \frac{75}{70} - 1 = \frac{5}{70}}$$

(e) Determine d_6: $a(6)(1-d_6) = a(5)$

$$\Rightarrow \boxed{d_6 = 1 - \frac{(.75)^{-1}}{(.7)^{-1}} = 1 - \frac{70}{75} = \frac{5}{75}}$$

2) $d = .06$ simple $\Longrightarrow$ $a(t) = (1 - .06t)^{-1}$ t-years



$$X = 1000 \, \frac{a(5)}{a(4)}$$

$a(4) = (.76)^{-1}$ $\quad a(5) = (.7)^{-1}$

$$\therefore \; X = 1000 \, \frac{(.7)^{-1}}{(.76)^{-1}} = 1000 \left(\frac{76}{70} \right) \Longrightarrow \boxed{X \doteq 1085.71}$$

3) $d^{(2)} = .08 \Longrightarrow d = \frac{d^{(2)}}{2} = \frac{.08}{2} = .04 = \text{sedr}$

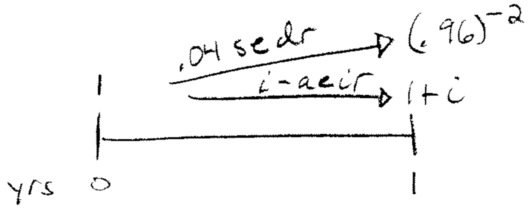
Since we're compounding, $d = .04$ = <u>the</u> semiannual effective discount rate (sedr). Unlike the "simple" discount scenario, when compounding, the edr is constant for every period. ($d_k = d$ = <u>the</u> periodic edr)

(a): $\boxed{a(t) = (1 - .04)^{-t} = (.96)^{-t}}$

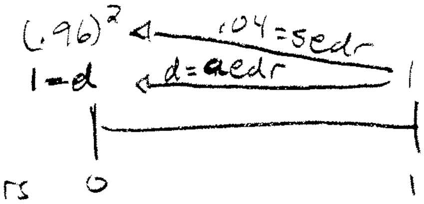
(b): $\boxed{v = 1 - d = .96 = \text{sdf}}$

$\boxed{v^{-1} = (.96)^{-1} = \text{saf}}$

(c):



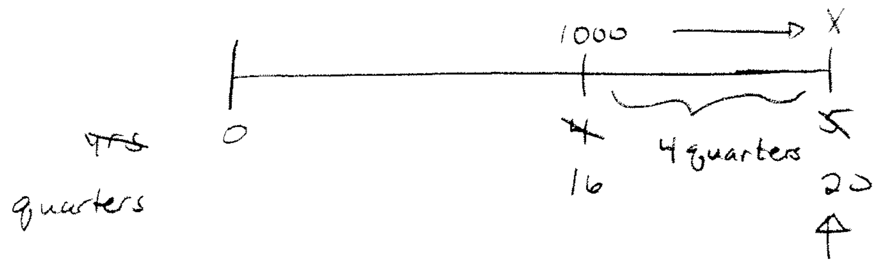
$\Longrightarrow \boxed{i = .96^{-2} - 1 = \text{aeir}}$



$\Longrightarrow \boxed{d = 1 - (.96)^2 = \text{aedr}}$

Note that $1 + i = (1-d)^{-1}$.

4) $d^{(4)} = .06 \Longrightarrow \frac{d^{(4)}}{4} = \frac{.06}{4} = .015 = qedr$



$qedr = .015 \Longrightarrow (1-.015)^{-1} = (.985)^{-1} = qaf$

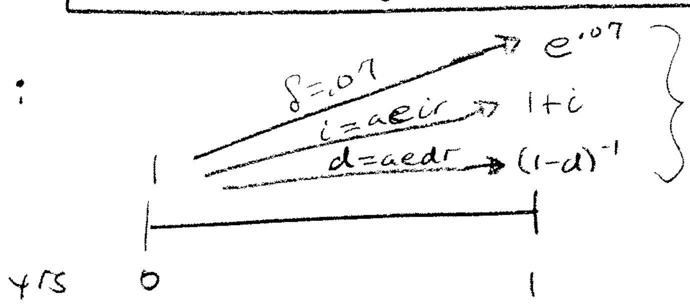
$\therefore \boxed{X = 1000(.985)^{-4} \doteq 1062.32}$

5) $\delta = .07$ (same as force of interest)

(a): $\boxed{a(t) = e^{.07t} \quad t\text{-years}}$

(b): $\boxed{e^{.07} = aaf \; ; \; adf = \frac{1}{e^{.07}} = e^{-.07} = v}$

(c):



$\Longrightarrow \boxed{\begin{array}{l} i = e^{.07} - 1 \\ d = 1 - e^{-.07} \end{array}}$

6) $d = .07 = aedr \Longrightarrow e^{\delta} = (1-d)^{-1} = (.93)^{-1} = aaf$

$\therefore \boxed{\delta = -\ln(.93)}$

7) $d^{(2)} = .04 \Longrightarrow \frac{.04}{2} = .02 = sedr$

$\Longrightarrow aaf = (1-d)^{-2} = (.98)^{-2} = e^{\delta}$

$\Longrightarrow \boxed{\delta = -2\ln(.98)}$

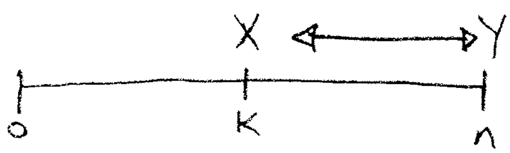
Section 4: General Force of Interest

Relating force of interest to accumulation functions:

Given $a(t)$, then $\delta_t = \frac{a'(t)}{a(t)}$ (t is measured in years)

Given δ_t, then $a(t) = e^{\int_0^t \delta_r \, dr}$ (t is measured in years)

Accumulating and Discounting using accumulation functions:



$$Y = X \cdot e^{\int_k^n \delta_t \, dt}, \text{ or equivalently, } X = Y \cdot e^{\int_n^k \delta_t \, dt}$$

Special Cases:

1.

$$\delta_t = c \cdot \frac{f'(t)}{f(t)} \Rightarrow a(t) = \left(\frac{f(t)}{f(0)}\right)^c$$

2. Constant Force of Interest: $\delta_t = \delta$ (see earlier notes on continuous compounding)

$$a(t) = e^{\delta t}$$

Module 1 Section 4 Problems:

1. Given $a(t) = 1 + 2t + \frac{1}{2}t^2$, determine an expression for the general force of interest.

2. Given $a(t) = 100 + 200t + 50t^2$, determine δ_2.

3. Given $\delta_t = \frac{6t}{2+6t^2}$ determine $a(1)$.

4. Suppose $\delta_t = .02t, t > 0$.
   a. Determine the accumulation function.
   b. Determine the accumulated value at time 7 of the time 3 value of 100.

5. Given $\delta_t = \frac{.03}{1-.03t}$ determine the discounted value at time 2 of the time 6 value of 50.

Solutions to Module 1 Section 4 Problems

1) $a(t) = 1 + 2t + \frac{1}{2}t^2 \implies a'(t) = 2 + t$

$$\therefore \boxed{\delta_t = \frac{a'(t)}{a(t)} = \frac{2+t}{1+2t+\frac{1}{2}t^2}}$$

2) Note that $a(0) = 100$. So the given function is an <u>amount function</u>, not an accumulation function. Our notation is $A(t) = 100 + 200t + 50t^2$. Factoring out 100, we get $A(t) = 100\underbrace{(1+2t+\frac{1}{2}t^2)}_{=\text{accumulation function}}$

Using our notation, we write $a(t) = 1 + 2t + \frac{1}{2}t^2$. This is the same as #1. So $\delta_t = \dfrac{2+t}{1+2t+\frac{1}{2}t^2}$

$$\implies \boxed{\delta_2 = \frac{4}{1+4+2} = \frac{4}{7}}$$

<u>Note</u>: Since $A(t) = C \cdot a(t)$, then $A'(t) = C \cdot a'(t)$.

$\therefore \delta_t = \dfrac{a'(t)}{a(t)} = \dfrac{A'(t)}{A(t)}$. Using this fact, we get

$$\delta_t = \frac{200+100t}{100+200t+50t^2} \implies \delta_2 = \frac{400}{100+400+200} = \frac{4}{7} \checkmark$$

3) $\delta_t = \dfrac{6t}{2+6t^2} = \dfrac{1}{2} \cdot \dfrac{12t}{2+6t^2}$ (This is the special case.)

$c = \frac{1}{2} \qquad f(t) = 2 + 6t^2 \implies f(0) = 2$

$$\therefore a(t) = \left[\frac{f(t)}{f(0)}\right]^c = \left[\frac{2+6t^2}{2}\right]^{1/2} = \sqrt{1+3t^2}$$

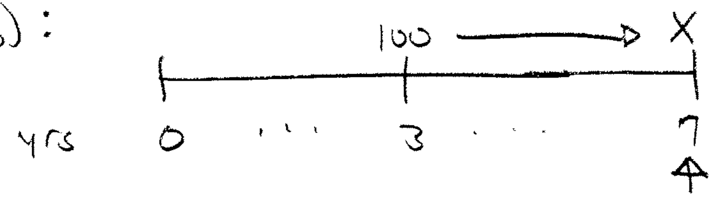
$$\implies \boxed{a(1) = \sqrt{4} = 2}$$

4) $\delta_t = .02t\ ,\ t > 0$

(a): $a(t) = e^{\int_0^t .02r\,dr} = e^{.01r^2\big|_0^t} = e^{.01t^2}$

$\boxed{a(t) = e^{.01t^2}}$

(b):



$X = 100\, e^{\int_3^7 .02t\,dt} = 100\, e^{.01t^2\big|_3^7}$

$\therefore \boxed{X = 100\, e^{0.4} \doteq 149.18}$

Alternatively,

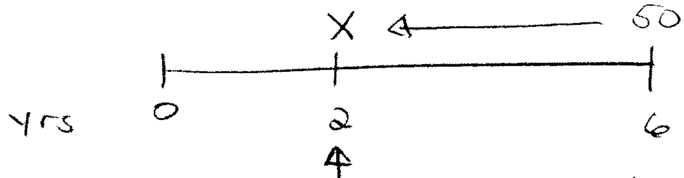
$X = 100\, \dfrac{a(7)}{a(3)} = 100\, \dfrac{e^{.49}}{e^{.09}} = 100\, e^{.4}$ ✓

5) $\delta_t = \dfrac{.03}{1-.03t} = (-1)\, \dfrac{-.03}{1-.03t}$ (special case)

$c = -1 \quad f(t) = 1 - .03t \quad f(0) = 1$

$\therefore \quad a(t) = \left[\dfrac{f(t)}{f(0)}\right]^c = \left(\dfrac{1-.03t}{1}\right)^{-1}$

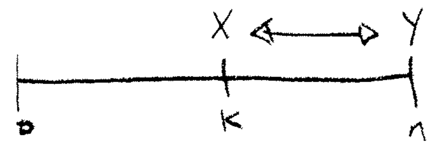
$\Rightarrow a(t) = (1-.03t)^{-1}$ (This is just simple discount with $d = .03$)



$X = 50\, \dfrac{a(2)}{a(6)} = 50\, \dfrac{(.94)^{-1}}{(.82)^{-1}} = 50\left(\dfrac{82}{94}\right)$

$\therefore \boxed{X \doteq 43.62}$

Section 5: Summary



$$Y = X \frac{a(n)}{a(k)}$$

$$X = Y \frac{a(k)}{a(n)}$$

Accumulation Functions:

Interest Scenario	Accumulation Function
i – simple interest	$a(t) = 1 + it$, t measured in years
i – periodic eir	$a(t) = (1+i)^t$, t measured in periods
d – simple discount	$a(t) = (1-dt)^{-1}$, t measured in years
d – periodic edr	$a(t) = (1-d)^{-t}$, t measured in periods
δ_t – general force of interest	$a(t) = e^{\int_0^t \delta_r\, dr}$, t measured in years

Constant Force of Interest Special Case: (Continuous Compounding)

$$\delta_t = \delta \Longrightarrow a(t) = e^{\delta t}$$

General Force of Interest Special Case:

$$\delta_t = c \cdot \frac{f'(t)}{f(t)} \Longrightarrow a(t) = \left(\frac{f(t)}{f(0)}\right)^c$$

Periodic Effective Interest and Discount Rates: (eir's and edr's)

$$i_k = \frac{a(k) - a(k-1)}{a(k-1)} \qquad d_k = \frac{a(k) - a(k-1)}{a(k)}$$

When compounding,

$$i_k = i = \frac{i^{(m)}}{m} \qquad d_k = d = \frac{d^{(m)}}{m}$$

periodic accumulation factor $= 1 + i = (1-d)^{-1}$

periodic discount factor $= v = 1 - d = (1+i)^{-1}$

Annual Compounding Case: (i = aeir and d = aedr and $\delta_t = \delta$)

$$v = 1 - d = (1+i)^{-1} = e^{-\delta}$$